Mathematics 272 Lecture 3 Notes

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1 Verifying Permutons as Limits and Integrating the Joint CDF of a Permuton

1.1 Recap of last lecture

Last time, we defined the **subpermutation density**

$$d(\sigma, \pi) = \text{prob. } |\sigma| \text{ random elements of } \pi \text{ yield } \sigma \text{ as a subpermutation},$$

where σ, π are two permutations. We also defined **permutons**, which are probability measures on the Borel σ-algebra of $[0,1]^2$ that have uniform marginals; these represent an analytic notion of permutations.

A **μ-random permutation of size** k, where μ is a permuton, is generated by sampling k points independently from μ and reading a permutation off of the ordering of x- and y-coordinates. We also defined $d(\sigma, \mu)$ to be the probability a μ-random permutation of size $|\sigma|$ is σ.

We said that a sequence $(\pi_n)_{n \in \mathbb{N}}$ of permutations is **convergent** if $|\pi_n| \to \infty$ and for every permutation σ, $\lim_{n \to \infty} d(\sigma, \pi_n)$ exists. A permuton μ is a limit of a convergent sequence $(\pi_n)_{n \in \mathbb{N}}$ if for every permutation σ, $\lim_{n \to \infty} d(\sigma, \pi_n) = d(\sigma, \mu)$. Last time, we wanted to prove the following statement:

Proposition 1.1. *Every convergent sequence of permutations has a limit permuton.*

The limit is unique, and proving the uniqueness is left as an exercise. Last time, we proved the following lemma:

Lemma 1.1. *Without loss of generality, for every k, there exists an n_0 such that for all $n \geq n_0$, 2^k divides $|\pi_n|$.*

We also defined matrices A_n^k, which were $2^k \times 2^k$ matrices, by

$$(A_n^k)_{i,j} := \frac{\{x : \frac{i-1}{2^k}|\pi_n| < x \leq \frac{i}{2^k}|\pi_n| \text{ and } \frac{j-1}{2^k}|\pi_n| < x \leq \frac{j}{2^k}|\pi_n|}{|\pi_n|},$$

where we index the matrices from left to right and bottom to top. We also defined A^k to be the coordinate-wise limits of A^k_n. Using these matrices, we defined a premeasure μ_0 on the dyadic squares in $[0,1]^2$ by $\mu_0([\frac{i-1}{2^k}, \frac{i}{2^k}) \times [\frac{j-1}{2^k}, \frac{j}{2^k})) = (A^k)_{i,j}$. Using Carathéodory's extension theorem, we furnished the existence of a measure μ on the Borel σ-algebra extending μ.

1.2 Verifying our permuton construction

Now, let's finish the proof.

Proof. We have a convergent sequence $(\pi_n)_{n\in\mathbb{N}}$ of permutations and a measure μ. and we want to show that for every permutation σ,

$$d(\sigma, \mu) = \lim_{n\to\infty} d(\sigma, \pi_n).$$

We will show that for every permutation σ and $\varepsilon > 0$, $|d(\sigma, \mu) - \lim_{n\to\infty} d(\sigma, \pi_n)| < \varepsilon$. Fix σ, and let k be such that $2^k > |\sigma|$. Let

$$\rho(\sigma, k) = \mathbb{P}(\mu\text{-random permutation of size } |\sigma| = \sigma \mid E),$$

where E is the event such that among the points $(x_1, y_1), \ldots, (x_{|\sigma|}, y_{|\sigma|})$, the x- or y-coordinates of no two points fall into the same dyadic interval. That is, there do not exist $1 \le i \ne j \le |\sigma|$ and $1 \le m \le 2^k$ such that $x_1 \in [\frac{m-1}{2^k}, \frac{m}{2^k})$ and $y_i \in [\frac{m-1}{2^k}, \frac{m}{2^k})$ and likewise for the y-coordinates. The idea is that we don't want to sample two points from the same row or the same column of our grid.

We can crudely upper bound

$$\mathbb{P}(E^c) \le 2\binom{|\sigma|}{2}\frac{1}{2^k}$$

using a union bound. As a consequence,

$$\rho(\sigma, k)\left(1 - 2\binom{|\sigma|}{2}\frac{1}{2^k}\right) \le d(\sigma, \mu) \le \rho(\sigma, k)\left(1 - 2\binom{|\sigma|}{2}\frac{1}{2^k}\right) + 2\binom{|\sigma|}{2}\frac{1}{2^k}.$$

So we conclude that

$$|d(\sigma, \mu) - \rho(\sigma, k)| \le 2\binom{|\sigma|}{2}\frac{1}{2^k}.$$

We will now argue that

$$\rho(\sigma, k)\left(1 - 2\binom{|\sigma|}{2}\frac{1}{2^k}\right) \le d(\sigma, \pi_n) \le \rho(\sigma, k)\left(1 - 2\binom{|\sigma|}{2}\frac{1}{2^k}\right) + 2\binom{|\sigma|}{2}\frac{1}{2^k},$$

and letting $n \to \infty$ will give us that

$$\left|\lim_{n\to\infty} d(\sigma, \pi_n) - \rho(\sigma, k)\right| \leq 2\binom{|\sigma|}{2}\frac{1}{2^k}.$$

Then we will have proven the desired claim with $\varepsilon = 4\binom{|\sigma|}{2}\frac{1}{2^k}$. Letting $k \to \infty$ will then finish the proof.

Let us observe that

$$\rho(\sigma, k) = \sum_{\substack{1\leq x_1<\cdots<x_{|\sigma|}\leq 2^k \\ 1\leq y_1<\cdots<y_{|\sigma|}\leq 2^k}} |\sigma|! \prod_{i=1}^{k-1}(A^k)_{x_i, y_{\sigma(i)}},$$

where the $|\sigma|!$ comes from all different orders we can sample the $|\sigma|$ points. If we define $\rho_n(\sigma, k)$ to be the same thing but with respect to π_n, then

$$\rho_n(\sigma, k)\left(1 - 2\binom{|\sigma|}{2}\frac{1}{2^k}\right) \leq d(\sigma, \pi_n) \leq \rho_n(\sigma, k)\left(1 - 2\binom{|\sigma|}{2}\frac{1}{2^k}\right) + 2\binom{|\sigma|}{2}\frac{1}{2^k},$$

and the desired bound follows. □

1.3 Integrating the joint CDF of a permuton

Theorem 1.1. *Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of permutation such that $\lim_{n\to\infty} d(\sigma, \pi_n) = \frac{1}{24}$ for every $\sigma \in S_4$. Then $(\pi_n)_{n\in\mathbb{N}}$ is convergent, and $\lim_{n\to\infty} d(\sigma, \pi_n) = \frac{1}{|\sigma|!}$ for every permutation σ.*

The statement we will show later (and make the preparations for today) is the following stronger version of the theroem.

Theorem 1.2. *Let μ be a permuton. If $d(\sigma, \mu) = \frac{1}{24}$ for every $\sigma \in S_4$, then μ is the uniform measure on $[0, 1]^2$.*

Fix a permuton μ, and denote λ to be the uniform measure. Define the joint CDF

$$F(x, y) := \mu([0, x] \times [0, y]).$$

We will see that it is useful to compute the integral

$$\int F(x, y)\, d\lambda = \mathbb{P}(X' \leq X, Y' \leq Y),$$

where $(X, Y) \sim \lambda$ and $(X', Y') \sim \mu$. If we were integrating with respect to μ, we would have

$$\int F(x, y)\, d\mu = \mathbb{P}(X' \leq X'', Y' \leq Y'') = \frac{1}{2}d(\ 1\ \ 2\ , \mu),$$

where $(X'', Y'') \sim \mu$.

With this as our motivation, we can rewrite the original integral as

$$\int F(x,y)\,d\lambda = \mathbb{P}(X' \leq X, Y' \leq Y),$$

where $(X, Z_x) \sim \mu$ and $(Y, Z_y) \sim \mu$. That is, we instead sample 3 pairs of points from μ. There are only 6 possibilities for the ordering of the points:

$$\begin{aligned} &= \frac{2}{6} d(\ 1 \quad 2 \quad 3\ , \mu) + \frac{2}{6} d(\ 1 \quad 3 \quad 2\ , \mu) + \frac{2}{6} d(\ 2 \quad 1 \quad 3\ , \mu) \\ &\quad + \frac{1}{6} d(\ 2 \quad 3 \quad 1\ , \mu) + \frac{1}{6} d(\ 3 \quad 1 \quad 2\ , \mu) + \frac{1}{6} d(\ 3 \quad 2 \quad 1\ , \mu). \end{aligned}$$

Next time, we will see how this helps us prove the theorem.